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Exact Solutions in Brans–Dicke Theory with Bulk Viscosity

Luis O. Pimentel[1]

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The effect of bulk viscosity on the evolution of the homogeneous and isotropic cosmological models in the Brans–Dicke theory of gravitation is studied. Solutions are found, with a baratropic equation of state, a time-independent bulk viscosity, the gravitational "constant" inversely proportional to the age of the universe, and the mass of the universe (in the closed model) proportional to the square of its age; the expansion factor is a linear function of the cosmological time. For flat space, power law expansions are found, among them one that is related to extended inflation.

1. INTRODUCTION

The renewed interest in the scalar–tensor theories of gravitation is caused by two main factors: First, most of the unified theories, including superstring theories, contain a scalar field (dialaton, size of the extra compact space in Kaluza–Klein theories) which play a similar role to the scalar field of the scalar–tensor theories. Second, the new scenario of extended inflation solves the fine-tuning problem of the old; new and chaotic inflation has a scalar field that slows the expansion rate of the universe, from exponential to polynomial, allowing the completion of the phase transition from the de Sitter phase to a radiation-dominated universe.

In this work I consider the influence of viscosity on the evolution of isotropic cosmological models in the scalar–tensor theory of Brans–Dicke. In the isotropic case the only compatible form of viscosity is bulk viscosity. The effect of viscosity on a general relativistic cosmological model has been considered by Johri and Sudarshan (1988).

[1]Departamento de Física, Universidad Autónoma Metropolitana, CP 09340 México D.F., México.

2. FIELD EQUATIONS

The field equations in the presence of bulk viscosity for the isotropic and homogeneous line element

$$ds^2 = c^2\,dt - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right] \tag{1}$$

in the scalar–tensor theory of gravitation of Brans–Dicke are

$$(\dot{\phi}a^3)^{\cdot} = \beta\left[\rho - \frac{3}{c^2}\left(p - 3\zeta\frac{\dot{a}}{a}\right)\right]a^3, \qquad \beta = 8\pi(3+2\omega)^{-1} \tag{2}$$

$$\frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi\rho}{3c^2\phi} - \frac{1}{c^2}\frac{\dot{\phi}}{\phi}\frac{\dot{a}}{a} + \frac{\omega}{6c^2}\left(\frac{\dot{\phi}}{\phi}\right)^2 \tag{3}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = \frac{9\zeta}{c^2}\left(\frac{\dot{a}}{a}\right)^2 \tag{4}$$

In the above system we have three equations for the five unknowns $a(t)$, $\phi(t)$, $p(t)$, $\rho(t)$, and $\zeta(t)$. It is clear that in order to solve the system we need additional information; our first assumption here is a barotropic equation of state,

$$p = \epsilon c^2\rho, \qquad -1 \le \epsilon \le 1 \tag{5}$$

with it the field equations are

$$(\dot{\phi}a^3)^{\cdot} = \beta\left[\rho(1-3\epsilon) + \frac{9\zeta}{c^2}\left(\frac{\dot{a}}{a}\right)\right] \tag{6}$$

$$\frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi\rho}{3c^2\phi} - \frac{1}{c^2}\frac{\dot{\phi}}{\phi}\frac{\dot{a}}{a} + \frac{\omega}{6c^2}\left(\frac{\dot{\phi}}{\phi}\right)^2 \tag{7}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(1+\epsilon)\rho = \frac{9\zeta}{c^2}\left(\frac{\dot{a}}{a}\right)^2 \tag{8}$$

The second assumption that we shall make here is related to the time behavior of the gravitational "constant." Dirac (1937, 1938), motivated by Eddington's analysis of the dimensionless "constants," had suggested that the gravitational "constant" decreases with the age t of the universe and proposed an inversely linear relation (Dirac's first hypothesis) that we take here,

$$\phi = bt, \qquad b = \text{const} \tag{9}$$

Dirac also postulated an increase of the mass M of the universe proportional to the square of its age (Dirac's second hypothesis). Generalizations of Einstein's equations that include a variable "gravitational constant," apart from the metric, have been motivated, at least in part, by Dirac's hypotheses. Therefore, it seems interesting to explore them in the case of the scalar–tensor theory of Brans–Dicke with bulk viscosity. We shall return to this hypothesis after finding exact solutions.

Substituting equation (9) into equations (6)–(8), we have

$$3b\left(\frac{\dot{a}}{a}\right)=\beta\left[\rho(1-3\epsilon)+\frac{9\zeta}{c^2}\left(\frac{\dot{a}}{a}\right)\right] \tag{10}$$

$$\frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2+\frac{k}{a^2}=\frac{8\pi\rho}{3c^2bt}-\frac{1}{c^2t}\frac{\dot{a}}{a}+\frac{\omega}{6c^2t^2} \tag{11}$$

$$\dot{\rho}+3\frac{\dot{a}}{a}(1+\epsilon)\rho=\frac{9\zeta}{c^2}\left(\frac{\dot{a}}{a}\right)^2 \tag{12}$$

In the above system, we notice from the first equation that if we have a radiation-dominated universe, the bulk viscosity is a constant different from zero ($\zeta_0 = bc^2/3\beta$), in contrast with what happens in general relativity (Weinberg, 1972, p. 568). From equation (11) we have that the density is given by

$$\rho=\frac{3b}{8\pi}\left[\left(\frac{\dot{a}}{a}\right)^2bt+\frac{kc^2}{a^2}+\frac{1}{t}\frac{\dot{a}}{a}+\frac{\omega}{6t^2}\right] \tag{13}$$

Substituting this expression for the density in equation (10), we obtain

$$3b\left(\frac{\dot{a}}{a}\right)=\beta\left[\frac{3b}{8\pi}(1-3\epsilon)\left(\left(\frac{\dot{a}}{a}\right)^2+\frac{kc^2}{a^2}+\frac{1}{t}\left(\frac{\dot{a}}{a}\right)+\frac{\omega}{6t^2}\right)+\frac{9\zeta}{c^2}\left(\frac{\dot{a}}{a}\right)\right] \tag{14}$$

3. $k \neq 0$ SOLUTIONS

By inspection we can see that equation (14), for $k \neq 0$, has the solution

$$a=dt, \qquad \zeta=\zeta_0=\text{const} \tag{15}$$

where d is a constant. Substituting the solution given by equation (15) into equation (13), we obtain the energy density,

$$\rho=\frac{q}{t}, \qquad q=\frac{3b}{8\pi}\left(2+\frac{kc^2}{d^2}-\frac{\omega}{6}\right) \tag{16}$$

The values of the constants d, ζ_0, and q in terms of k, w, and b (this last related to G_0) are obtained after substituting equation (16) into equations (10)–(12)

$$d^2=\frac{6c^2k}{5\omega-6} \tag{17}$$

$$\zeta_0=\frac{bc^2(2+3\epsilon)(3+2\omega)}{72\pi} \tag{18}$$

$$q=\frac{b(3+2\omega)}{8\pi} \tag{19}$$

We recall that these solutions are valid only for $k = \pm 1$, but if we take $k = -1$, then we need $\omega < 6/5$ to have a real d; on the other hand, equation (19) requires $\omega > -3/2$ to have a positive energy density. This leaves us with a range for the values of ω that is in conflict with the observations. For $k = 1$ we do not have that problem. For the closed space, the mass of the universe has a time dependence that is given by

$$M = 2\pi^2 \rho a^3 = 2\pi^2 q\, d^3 t^2 \tag{20}$$

that is, the second Dirac hypothesis is satisfied by the solution for nonflat space. In contrast with the solutions obtained in general relativity (Padmanabhan and Chitre, 1987), here a constant bulk viscosity does not lead to an inflationary phase, since the type of expansion is not determined by the magnitude of the bulk viscosity.

Recalling the relation for ϕ at the present time (Weinberg, 1972)

$$\phi_0 = G_0^{-1} \frac{4 + 2\omega}{3 + 2\omega} \tag{21}$$

and the definition of the Hubble constant,

$$H_0 = \left(\frac{\dot{a}}{a}\right)_0 = \frac{1}{t_0} \tag{22}$$

(here t_0 is the age of the universe), we obtain the value of the constant b,

$$b = \frac{H_0}{G_0} \frac{4 + 2\omega}{3 + 2\omega} \tag{23}$$

Now the present values of the density and the bulk viscosity can be calculated,

$$\rho_0 = \frac{H_0^2}{G_0} \frac{4 + 2\omega}{8\pi} = \frac{4 + 2\omega}{3} \rho_c \tag{24}$$

$$\zeta_0 = \frac{c^2 H_0}{G_0} \frac{(2 + 3\epsilon)(4 + 2\omega)}{72\pi} \tag{25}$$

Here $\rho_c = 3H_0^2/8\pi G_0$ is the critical density.

Another property of the above solutions is

$$\frac{MG}{c^2 a} = \text{const} \tag{26}$$

Universes satisfying this relation are called sometimes "Machian."

4. $k = 0$ SOLUTIONS

For this case by inspection also we see that a solution is given by

$$a = dt^n, \qquad \zeta = \text{const} \tag{27}$$

where d and n are constants. Substituting the solution given by equation (27) into equation (13), we obtain the energy density,

$$\rho = \frac{q}{t}, \qquad q = \frac{3b}{8\pi}\left(n^2 + n - \frac{\omega}{6}\right) \tag{28}$$

Using equation (22) in equations (10) and (12) and eliminating ζ, we obtain the following cubic equation for n:

$$(2n + 1)(6n - 12n^2 - \omega + 6n\omega) = 0 \tag{29}$$

Each one of the solutions to this equation will give a different solution with different values for ζ and q; they are

$$n_1 = -1/2, \tag{30}$$

$$\zeta_1 = \frac{bc^2(5 + 3\epsilon)(3 + 2\omega)}{144\pi} \tag{31}$$

$$q_1 = \frac{-b(3 + 2\omega)}{32\pi} \tag{32}$$

$$n_2 = \frac{3 + 3\omega + \sqrt{f(\omega)}}{12} \tag{33}$$

$$\zeta_2 = \frac{bc^2[-1 + 3\epsilon + 3\omega + 3\epsilon\omega + (1 + \epsilon)\sqrt{f(\omega)}][9 + 6\omega + 3\omega^2 + (3 + \omega)\sqrt{f(\omega)}]}{16\pi[3 + 3\omega + \sqrt{f(\omega)}]^2} \tag{34}$$

$$q_2 = \frac{b[9 + 6\omega + 3\omega^2 + (3 + \omega)\sqrt{f(\omega)}}{64\pi} \tag{35}$$

$$n_3 = \frac{3 + 3\omega - \sqrt{f(\omega)}}{12} \tag{36}$$

$$\zeta_3 = \frac{bc^2[-1 + 3\epsilon + 3\omega + 3\epsilon\omega - (1 + \epsilon)\sqrt{f(\omega)}][9 + 6\omega + 3\omega^2 - (3 + \omega)\sqrt{f(\omega)}]}{16\pi[3 + 3\omega - \sqrt{f(\omega)}]^2} \tag{37}$$

$$q_3 = \frac{bc^2[9 + 6\omega + 3\omega^2 - (3 + \omega)\sqrt{f(\omega)}}{64\pi} \tag{38}$$

where

$$f(\omega) = 9 + 6\omega + 9\omega^2 \tag{39}$$

The value of d is not determined as usual in the case of flat space. From the three solutions we can discard the first one because it is not expanding. Since $\omega \geq 500$, $n_2 \approx w/2$ and $n_3 \approx 0$. Therefore, the second solution represents an extended inflationary phase produced by the bulk viscosity.

All the solutions obtained in this paper have a constant bulk viscosity; therefore they can model some stage of the universe in which the viscosity can be considered constant, or if they are taken to be valid for all the history of the universe, the constant viscosity should be small enough so that at present times it is consistent with the observations.

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